Algebraic Geometry Lecture 5 – Local Rings of Varieties

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Def$^{\mathbf{n}}$. A variety over $k = \overline{k}$ is any affine, quasi-affine, projective, or quasi-projective variety.

Notation: $k[X] = k[X_1, \ldots, X_n]$.

Recall, the coordinate ring $k[V]$ of an affine variety V is

$$k[V] = \{f : V \to k \mid f \text{ is a polynomial function}\} \cong k[X]/I(V).$$

Def$^{\mathbf{n}}$. $f : V \to k$ is regular at $P \in V$ if there exists an open neighbourhood U with $P \in U \subseteq V$ and there exist $g, h \in k[X]$ such that $f = g/h$ on U and $h(Q) \neq 0$ for all $Q \in U$.

Lemma. *A regular function on a (quasi-) affine variety is continuous in the Zariski topology (where k is identified with $\mathbb{A}^1$).*

Proof. We need to show $f^{-1}(S)$ is open for all open sets S. Equivalently that $f^{-1}(T)$ is closed for all closed sets T.

A closed set in $k(= \mathbb{A}^1)$ is a finite collection of points. So if we can show the preimage of a point is closed then we're done, as the union is finite. This would be easy if we could write $f = g/h$ everywhere, but we can't. We'll use a lemma from topology: a set $T \subset \mathbb{A}^1$ is closed iff T can be covered by open subsets U such that $T \cap U$ is closed in U for each U.

Let U be an open set on which $f = g/h$ for $g, h \in k[X]$. Then

$$f^{-1}(a) \cap U = \{P \in U \mid g(P)/h(P) = a\}.$$

But $g(P)/h(P) = a$ iff $(g - ah)(P) = 0$. So

$$f^{-1}(a) \cap U = Z(g - ah) \cap U,$$

where $Z(\cdot)$ is the zero set, our "V" from previous lectures. This is closed by definition, hence $f^{-1}(a)$ is closed, thus so is $f^{-1}(T)$. □

We can also prove this with projective varieties.

We use $\mathcal{O}(V)$ to mean the regular functions on V.

[1]Notes typed by Lee Butler based on a lecture given by Joe Grant. Any errors are the responsibility of the typist. Or Andrew Potter, the scoundrel.

Def$^{\mathbf{n}}$. Let V be a variety and P be a point in V. Define the local ring of V at P, denoted $\mathcal{O}_{P,V}$ to be

$$\mathcal{O}_{P,V} = \{f : U \to k \mid U \subset V \text{ is open, } P \in U, f \text{ is regular on } U\}\Big/\sim$$

where $f : U \to k \sim g : W \to k$ iff $f = g$ on $U \cap W$. (If V is clear from the context then we write $\mathcal{O}_{P,V} = \mathcal{O}_P$.)

We require $\sim$ to be an equivalence relation. It is clearly reflexive and symmetric, so we just need to check transitivity. Suppose $f \sim g$ and $g \sim h$ with $f : U \to k$, $g : W \to k$, $h : X \to k$. Then $f = g$ on $U \cap W$ and $g = h$ on $W \cap X$. We want to show thath $f = h$ on $U \cap X$. Note that as $U \cap W \cap X \subseteq \begin{cases} U \cap W \\ W \cap X \end{cases}$ we have $f = h$ on $U \cap W \cap X$. So $f - h = 0$ on $U \cap W \cap X$. $\{0\} \in Z(x)$ is closed in $\mathbb{A}^1$ so by the lemma $(f-h)^{-1}(0)$ is a closed set in $U \cap W \cap X$. As W is open in V it is dense in V so $(f-h)^{-1}(0)$ is dense in $U \cap X$. But $(f-h)^{-1}(0)$ is closed so $(f-h)^{-1}(0) = U \cap X$. So $(f-h)(U \cap X) = 0$, so $f = h$ on $U \cap X$.

$\mathcal{O}_P$ is a commutative ring, we define addition on open intersections. So $\mathcal{O}_P/m = k$ where m is a maximal ideal. Our choice of m is the set of equivalence classes of regular functions vanishing at P. (Unique up to isomorphism.)

Def$^{\mathbf{n}}$. We define the function field $k(V)$ to be

$$k(V) = \{f : U \to k \mid U \subset V \text{ is an open subset, } f \text{ is regular on } U\}\Big/\sim$$

with $\sim$ as before. We call elements of $k(V)$ rational functions on V.

Note that $k(V)$ is a field: as V is irreducible any two non-empty open subsets have a non-empty intersection. So we can define addition and multiplication on them. Also, if f is defined on U then we can define $1/f$ on $U \setminus (U \cap Z(f))$ on which $1/f$ is regular.

So we have, by restriction of functions,

$$\mathcal{O}(V) \hookrightarrow \mathcal{O}_P \hookrightarrow k(V),$$

for $P \in V$.

These are invariants of V, P. We will relate these to the affine coordinate ring $k[V] \cong k[X]/I(V)$.

Theorem. (a) *For each $P \in V$ if we let $\mathfrak{m}_P$ be the ideal of functions vanishing at P, then $P \mapsto \mathfrak{m}_P$ gives a 1-1 correspondence between points of V and maximal ideals of $k[V]$.*
(b) *For each $P \in V$, $\mathcal{O}_P \cong k[V]_{\mathfrak{m}_P}$ where $k[V]_{\mathfrak{m}_P}$ is the localisation of $k[V]$ at $\mathfrak{m}_P$, essentially assigning formal inverses.*
(c) *$k(V)$ is isomorphic to quotient fields of $k[V]$.*
(d) *$\mathcal{O}(V) \cong k[V]$.*

In the projective case we need various modifications. For Y a projective variety with coordinate ring $S(Y)$,

(a) $\mathcal{O}(Y) \cong k$. (Different to $\mathbb{A}$!)
(b) $\mathcal{O}_P \cong S(Y)_{\mathfrak{m}_P}$. (Similar.)
(c) $k(Y) \cong S(Y)_(0)$. (Similar.)